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FORCED PERIODIC MOTIONS OF A QUASIPERIODIC SYSTEM WITH A LAG*

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The forced periodic motions of a qusilinear oscillator are investigated using Poincaré's method and the method of successive approximations /1/. It is assumed that the perturbing function contains variables with deflecting arguments. Sufficient conditions of asymptotic stability are obtained by the exponential law of derived periodic motions, using Lyapunov's first method /1/ and by applying the Floquet-Lyapunov theory of the differential equations with periodic coefficients /2, 3/. Specific examples of perturbations, in particular, the analogue of Duffing's equation are considered. These investigations may be useful when considering problems of the control of oscillatory and rotating systems, using small controlling actions with an significant time lag. This lag is usually generated by the finite velocity of transmission of various signals in the control system, by the time taken to process measurement data, and the inertia of actuating mechanisms /4/. In systems containing distributed elements, the lag is due to the finite propagation velocity of wave processes, defined by the properties of the medium /5/, etc. Small-parameter methods were used by Krasovskii /6/, Shimanov /2, 7/, and others (see the bibliography in /3, 5/) to investigate oscillating systems with a time lag.

1. Statement of the problem.

We consider the quasilinear system

$$z^{\cdot\cdot} + \omega^2 z = G(t) + \varepsilon g(t, z, z^{\cdot}, z_\tau, z_\tau^{\cdot}), \quad |t| < \infty \tag{1.1}$$

where z is a scalar variable $z = z(t)$, $z_\tau = z(t-\tau)$, the dots denote derivatives with respect to time t, ω are parameters of the frequency of unperturbed oscillations (the case when $\omega = 0$ is also considered), τ is the deviation of the argument, $|\tau| < \infty$, and ε is a small parameter $\varepsilon \in [0, \varepsilon_0]$. The functions G and g are assumed to be piecewise smooth with respect to t, T_ν-periodic ($T_\nu = 2\pi/\nu$, $\nu > 0$), and may admit in the interval $t \in [t_0, t_0 + T_\nu]$ a finite number of points of discontinuity of the first kind. When $\varepsilon = 0$ the generating system is presumed to have a solution z_0 that is T-periodic which can be represented by the trigonometric series

$$z_0 = \sum_{k=-\infty}^{\infty} \frac{G^{(k)}}{\omega^2 - k^2\nu^2} e^{ik\nu t}, \qquad G(t) \sim \sum_{k=-\infty}^{\infty} G^{(k)} e^{ik\nu t} \tag{1.2}$$

When deriving a particular solution of (1.2), it is required that $\omega \neq k\nu$ or $G^{(k)} = 0$ when $\omega = k\nu$. We reduce (1.1) by the substitution $z = z_0 + x$ to the form

$$x^{\cdot\cdot} + \omega^2 x = \varepsilon f(t, x, x^{\cdot}, x_\tau, x_\tau^{\cdot}), \quad |t| < \infty \tag{1.3}$$

The function f is known and is T_ν-periodic in t. It is simply derived on the basis of the function g and by putting $z = z_0 + x$, and is considered to be piecewise smooth in t and fairly smooth relative to other arguments in some region of its determination. The property of smoothness and other properties of system (1.3) are defined more exactly below.

We have the problem of deriving a perturbed periodic solution for system (1.3) with deflecting argument for any $\varepsilon \in [0, \varepsilon_0]$ for ε_0 fairly small, and of investigating its stability as $t \to \pm\infty$.

The periodic solution of (1.3), whether oscillatory or rotational (when $\omega = 0$) may be derived by Poincaré's constructive methods or by using successive approximations /1, 8/. Note that the lag or lead of the arguments are not distinguished in the derivation of T-periodic solutions. The initial function is not specified, but is determined when solving the boundary value problem. The solution system (1.3) is derived in some interval of time $t \in [t_0, t_0 + T]$ and continued in a periodically smooth manner for all $t > t_0 + T$ and $t < t_0$. The stability of periodic motions is investigated in the case of a lagging argument: $t \in [t_0, \infty)$, $\tau > 0$ or $t \in (-\infty, t_0]$, $\tau < 0$; the motions of a system with leading argument are unstable.

The so-called simple cases such as, when the number of critical characteristic exponents and their respective periodic solutions are the same (/1–3, /5–7/ etc.), are usually considered

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in investigations of stability. This occurs in both the resonance and non-resonance cases in system (1.3) when $\omega > 0$; $\omega = 0$ belongs to a special critical case.

Below, we consider the forced periodic motions of system (1.3) in the following cases: non-resonance oscillations ($\omega \neq (n/m)\,\nu$, where n, m are prime integers, resonance oscillations ($\omega = (n/m)\,\nu$), and special oscillations or rotations ($\omega = 0$).

2. Oscillations in the non-resonant case. *Derivation of the solution.* The T_ν-periodic solution $x = x(t, \varepsilon)$ is derived in the form $x = \varepsilon y$, where the unknown function y is determined from the equation ($\varepsilon > 0$)

$$y^{\cdot\cdot} + \omega^2 y = f(t, \varepsilon y, \varepsilon y^{\cdot}, \varepsilon y_\tau, \varepsilon y_\tau^{\cdot}) \tag{2.1}$$

If the function f is analytic with respect to $x, x^{\cdot}, x_\tau, x_\tau^{\cdot}$ in a small neighbourhood of the point $x = x^{\cdot} = x_\tau = x_\tau^{\cdot} = 0$, the solution $y = y(t, \varepsilon)$ is derived by expansions in powers of the small parameter ε

$$y = y_0 + \varepsilon y_1 + \ldots, \qquad y_p(t) = \sum_{k=-\infty}^{\infty} \frac{f_p^{(k)} e^{ik\nu t}}{\omega^2 - k^2\nu^2}, \qquad p = 0, 1, \ldots \tag{2.2}$$

where $f_p^{(k)}$ are the coefficients of the Fourier functions $f_p(t)$ determined successively, for example,

$$f_0(t) = f(t, 0, 0, 0, 0) \tag{2.3}$$
$$f_1(t) = (f_x')_0\, y_0 + (f_{x^{\cdot}}')_0\, y_0^{\cdot} + (f_{x_\tau}')_0\, y_{0\tau} + (f_{x_\tau^{\cdot}}')_0\, y_{0\tau}^{\cdot}$$

Expansions (2.2) and (2.3) do not have singularities, since we know a fortiori that $\omega \neq k\nu$. When $\varepsilon > 0$ is fairly small, they converge to the unique T_ν-periodic solution of (2.1) established by the method of majorizing series.

If the function f is non-analytic in $x, x^{\cdot}, x_\tau, x_\tau^{\cdot}$, the solution derived by the method of successive approximations using the recurrent scheme ($p = 0, 1, \ldots, y_{(0)} \equiv y_0$)

$$y_{(p+1)}(t, \varepsilon) = \sum_{k=-\infty}^{\infty} \frac{f_{(p+1)}^{(k)} e^{ik\nu t}}{\omega^2 - k^2\nu^2}, \qquad f_{(p+1)}(t) = f(t, \varepsilon y_{(p)}, \varepsilon y_{(p)}^{\cdot}, \varepsilon y_{(p)\tau}, \varepsilon y_{(p)\tau}^{\cdot}) \tag{2.4}$$

The successive approximations (2.4) uniformly converge to the unique T_ν-periodic solution of (2.1) with fairly small $\varepsilon > 0$, if the function f satisfies the Lipschitz condition in $x, x^{\cdot}, x_\tau, x_\tau^{\cdot}$ with constants independent of t in a small neighbourhood of the zero solution generation. This is established by Schauder's principle and the theory of the Banach compression operator /9/. Rigorous confirmation by perturbation methods are formulated and proved as in /6, 7/. The estimates of the radius of convergence of series (2.2) in ε, $|\varepsilon| \leqslant \varepsilon_0$ or the successive approximations (2.4) are obtained by conventional methods /1, 7/.

Investigation of stability. The analysis of the stability of the derived periodic solution (for respective definitions see /3, 5/) is based on the calculation of the critical characteristic exponents of the linear equaiton with periodic coefficients in variations /2/

$$\xi^{\cdot\cdot} + \omega^2\xi = \varepsilon(f_x'\xi + f_{x^{\cdot}}'\xi^{\cdot} + f_{x_\tau}'\xi_\tau + f_{x_\tau^{\cdot}}'\xi_\tau^{\cdot}) \tag{2.5}$$

where ξ is the variation of solution, and the derivatives of the function f are taken on the derived periodic solution $x = \varepsilon y(t, \varepsilon)$. The case considered here is a critical one: when $\varepsilon = 0$, Eq.(2.5) has a pair of purely imaginary roots $\lambda_{1,2} = \pm i\omega$.

As shown in /3/, it can be readily established using simple examples of (2.5) that when $|\varepsilon| > 0$, as $t \to +\infty$ for $\tau < 0$ or as $t \to -\infty$ for $\tau > 0$ (the cases of a leading argument), the motion is highly unstable. The cases of a lagging argument $\tau > 0$, $t \to +\infty$ or $\tau < 0$, $t \to -\infty$ require additional investigation involving the calculation of two critical characteristic exponents $\lambda_{1,2}$ taking ε into account. To substantiate the method used, results similar to those of Floquet's theory are used here /2/, namely, for differential equations that are linear, homogeneous, and T_ν-periodic with a Lagging argument (to be specific, we subsequently assume $t \geqslant 0, \tau \geqslant 0$) each solution can be approximated with any degree of accuracy on a scale of exponential functions by a linear combination of solutions of the form $t^k e^{\lambda_r t} u(t)$, $k = 0, 1, \ldots, k_r$, where $u(t + T_\nu) \equiv u(t)$. The constants λ_r have the meaning of characteristic exponents.

According to this proposition the solution of the variational equation (2.5) is derived in the form of the expansions

$$\xi = e^{\lambda t}u, \qquad u(t + T_\nu) \equiv u(t), \qquad \xi_\tau = e^{\lambda(t-\tau)}u_\tau \tag{2.6}$$
$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2 \ldots, \quad u = u_0 + \varepsilon u_1 + \varepsilon^2 \ldots$$

Substituting (2.6) into (2.5) and equating coefficients of like powers of ε, we obtain the required expressions for $\lambda_0, u_0, \lambda_1, u_1, \ldots$, and in particular $\lambda_0 = \pm i\omega$, $u_0 = c_1^0 + c_2^0 e^{-2\lambda_0 t}$. It follows from the condition of T_ν-periodicity of u_0 that $c_2^0 = 0$. From the condition of T_ν-periodicity of u_1 we determine λ_1

$$\lambda_1 = \frac{1}{2T_\nu}\int_0^{T_\nu}\left[\frac{1}{\lambda_0}(f_x')_0 + (f_{x^\cdot}')_0 + \frac{1}{\lambda_0}(f_{x_\tau}')e^{-\lambda_0\tau} + (f_{x_\tau^\cdot}')_0 e^{-\lambda_0\tau}\right]dt \tag{2.7}$$

For the real part ρ_1 and the imaginary part μ_1, respectively, of the exponent λ_1 (2.7) we obtain simple explicit expressions, which may be calculated using the functions x_0 $(x_0 \equiv 0)$

$$\rho_1 = \frac{1}{2T_\nu}\int_0^{T_\nu}\left[(f_{x^\cdot}')_0 - \frac{1}{\omega}(f_{x_\tau}')_0\sin\omega\tau + (f_{x_\tau^\cdot}')_0\cos\omega\tau\right]dt \tag{2.8}$$

$$\mu_1 = \frac{\mp 1}{2T_\nu}\int_0^{T_\nu}\left[\frac{1}{\omega}(f_x')_0 + \frac{1}{\omega}(f_{x_\tau}')_0\cos\omega\tau + (f_{x_\tau^\cdot}')_0\sin\omega\tau\right]dt$$

Hence the following statement holds.

Statement 1. The T_ν-periodic solution $x(t, \varepsilon)$ of a system with lagging argument (1.3) is asymptotically stable when $\varepsilon > 0$ is fairly small, if $\rho_1 < 0$, and unstable when $\rho_1 > 0$.

The case of $\rho_1 = 0$ requires additional investigation taking higher powers of ε into account; it can be carried out by expansions in series or by successive approximations of the exponent λ /2/ in ε. Note that when $\tau = 0$ (a system without lag), or when the averages of $(f_{x_\tau}')_0$, $(f_{x^\cdot_\tau}')_0$ are zero, Eqs.(2.8) is identical with those for systems without delay /1/. It follows from (2.8) that the case of "positional" perturbations $f = \varphi(t) + kx_\tau$ leads to asymptotically stable oscillations, when $k\sin\omega\tau > 0$, for example $k > 0$, $0 < \tau < \pi/\omega$. Conversely, even when there is linear dissociation $((f_{x^\cdot})_0 = b < 0)$, the stability of motion may, according to (2.8), be disrupted by the terms dependent on the lag.

3. Investigation of resonance oscillations. *Derivation of the solution.* System (1.3) is considered in the resonance case in which $\omega = (n/m)\nu$, where n, m are prime integers, and T is the periodic solution of period $T = mT_\nu$ derived by expansion in series in powers of the parameter ε, or by successive approximations in the form (see /2–3, 5–7/)

$$x(t, \varepsilon) = x_0 + \varepsilon y(t, \varepsilon), \quad x_0 = a\sin\omega t + b\cos\omega t \tag{3.1}$$

where a and b are constants to be determined, and y is an unknown T-periodic function which satisfies the equation

$$y^{\cdot\cdot} + \omega^2 y = f(t, x_0, x_0^\cdot, x_{0\tau}, x_{0\tau}^\cdot) + \varepsilon[(f_x')_0 y + (f_{x^\cdot}')_0 y^\cdot + (f_{x_\tau}')_0 y_\tau + (f_{x_\tau^\cdot}')_0 y_\tau^\cdot] + \varepsilon^2 R(t, y, y^\cdot, y_\tau, y_\tau^\cdot, \varepsilon) \tag{3.2}$$

It is assumed that the function R satisfies the Lipschitz condition on $y, y^\cdot, y_\tau, y_\tau^\cdot$ with constant independent of t in the small neighbourhood of the generating solution. Neglecting terms $O(\varepsilon)$ in (3.2) we obtain the expansion

$$y_0 = \frac{1}{\omega}\int_0^t f_0(s, a, b)\sin\omega(t-s)\,ds + \alpha_0\sin\omega t + \beta_0\cos\omega t, \qquad y_0^\cdot = dy_0/dt \tag{3.3}$$

where f_0 is a known function, T periodic in t, of the parameters a and b. The required function y_0 will be T-periodic, if

$$P(a, b) \equiv y_0(T) - y_0(0) = 0, \; Q(a, b) \equiv y^\cdot(T) - y^\cdot(0) = 0 \tag{3.4}$$

Relations (3.4) are considered as equations in the unknowns a and b. Let us assume that system (3.4) has a real root a^*, b^*. Then function x_0 (3.1) is completely defined, and y_0 in (3.3) is correct except for parameters α_0, β_0. These parameters are determined by the conditions of periodicity of the following approximations y_1, which reduce to the form of the system

$$P_{a*}'\alpha_0 + P_{b*}'\beta_0 = -y_1^\circ(T), \qquad Q_{a*}'\alpha_0 + Q_{b*}'\beta_0 = -y_1^{\circ\cdot}(T) \tag{3.5}$$

$$y_1^\circ(t) \equiv \frac{1}{\omega}\int_0^t [(f_x')_0 y_0^\circ + (f_{x^\cdot}')_0 y_0^{\circ\cdot} + (f_{x_\tau}')_0 y_{0\tau}^\circ + (f_{x_\tau^\cdot}')_0 y_{0\tau}^{\circ\cdot}] \times \sin\omega(t-s)\,ds$$

Here $y_0^\circ(t)$ is the function y_0 in (3.3), when $\alpha_0 = \beta_0 = 0$. If a^*, b^* is a simple root of system (3.4), i.e.

$$\Delta(\tau) = \left|\frac{\partial(P, Q)}{\partial(a, b)}\right|_{a^*, b^*} \neq 0 \tag{3.6}$$

and this is assumed, then the linear system (3.5) is uniquely solvable for α_0, β_0, and by the same token the function $y_0(t)$ is completely defined. Subsequent approximations $y_{(l)}$ are determined by the recurrent scheme $(l = 1, 2, \ldots)$

$$y_{(l+1)} = y_0^\circ(t) + \alpha_{l+1}\sin\omega t + \beta_{l+1}\cos\omega t + \varepsilon y_{(l)}^*(t, \alpha_l, \beta_l, \varepsilon) \tag{3.7}$$

$$y^*_{(l)} \equiv \frac{1}{\omega}\int_0^t [(f_x')_0 y_{(l)} + (f_{x^\cdot}')_0 y^\cdot_{(l)} + (f_{x_\tau}')_0 y_{\tau(l)} + (f_{x_\tau^\cdot}')_0 y^\cdot_{\tau(l)} + \varepsilon R(s, y_{(l)}, y^\cdot_{(l)}, y_{\tau(l)}, y^\cdot_{\tau(l)}, \varepsilon)] \sin\omega(t-s)\, ds$$

and for $l = 1$ we assume $R \equiv 0$. The parameters α_l, β_l, as functions of ε and of other specified parameters of the system, are obtained at each step by solving the quasilinear system of equations

$$P_{a*}'\alpha_l + P_{b*}'\beta_l = -y_1^\circ(T) + \varepsilon p_l(T, \alpha_l, \beta_l, \varepsilon) \tag{3.8}$$

$$Q_{a*}'\alpha_l + Q_{b*}'\beta_l = -y_1^{\circ\cdot}(T) + \varepsilon p_l^\cdot(T, \alpha_l, \beta_l, \varepsilon)$$

$$p_l(t, \alpha_l, \beta_l, \varepsilon) \equiv -\frac{1}{\omega}\int_0^t [(f_x')_0 y^*_{(l-1)} + (f_{x^\cdot}')_0 y^{*\cdot}_{(l-1)}) +$$

$$(f_{x_\tau}')_0 y^*_{\tau(l-1)} + (f_{x_\tau^\cdot}')_0 y^{*\cdot}_{\tau(l-1)} + R(s, y_{(l)}, y^\cdot_{(l)}, y_{\tau(l)}, y^\cdot_{\tau(l)}, \varepsilon)] \sin\omega(t-s)\, ds$$

Since the functions $p_l, p_l^\cdot$ satisfy the Lipschitz conditions on α_l, β_l in a small neighbourhood of α_0, β_0, system (3.8) has a unique root $\alpha_l(\varepsilon), \beta_l(\varepsilon)$ which when $\varepsilon = 0$ becomes α_0, β_0. As a result, we have a recurrent scheme for successive approximations (3.7), (3.8). Using the general theorems in /9/ and the investigations in /6, 7 and 3, 5/, we can establish that the limit as $l \to \infty$ is the unique T-periodic solution of system (3.2) $y(t, \varepsilon)$, and in conformity with the change (3.1), the function $x(t, \varepsilon)$ is the required (n/m)-resonance solution of system (1.3). A power convergence of approximations $x_{(l)} = x_0 + \varepsilon y_{(l)}$ to the required T-periodic solution $x^*(t, \varepsilon)$ then occurs. When the function f is analytic, the solution is derived by expansions similar to (2.2)

Conditions of stability. The asymptotic stability is investigated using the variational equation as in Sect.2 of /2/. The non-trivial solution and the critical characteristic exponents are sought in the form

$$\lambda = \varepsilon\lambda_1 + \varepsilon^2 \ldots, \quad u = u_0 + \varepsilon u_1 + \varepsilon^2 \ldots, \quad u(t + T) \equiv u(t)$$

For u_0 we obtain the expression $u_0 = A\sin\omega t + B\cos\omega t$, and the T-periodic function u_1 is obtained from the equation

$$u_1^{\cdot\cdot} + \omega^2 u_1 = -2\lambda_1 u_0^\cdot + (f_x')_0 u_0 + (f_{x_\tau}')_0 u_{\tau 0} + (f_{x^\cdot}')_0 u_0^\cdot + (f_{x^\cdot_\tau}')_0 u_{\tau 0}$$

The conditions of periodicity of u_1 reduce to the relations

$$\omega P_{a*}' A + (\omega P_{b*}' - \Lambda) B = 0 \tag{3.9}$$

$$(Q_{a*}' - \Lambda) A + Q_{b*}' B = 0, \quad \Lambda \equiv \lambda_1 T\omega$$

It follows from (3.9) that the determinant of the system must vanish for the required values of λ_1, i.e.

$$\Lambda^2 - \Lambda\delta(\tau) - \omega\Delta(\tau) = 0, \quad \delta = Q_{a*}' + \omega P_{b*}' \tag{3.10}$$

Analysis of the roots of (3.10) provides the following necessary and sufficient conditions for the real parts of both roots $\Lambda_{1,2}$ to be negative

$$\Delta(\tau) < 0, \quad \delta(\tau) < 0 \tag{3.11}$$

The determinant $\Delta(\tau)$ is calculated from (3.6) and is non-zero; the quantity $\delta(\tau)$ is similar to ρ_1 in (2.8) and is

$$\delta(\tau) = \int_0^T [\omega(f_{x^\cdot}')_0 - (f_x')_0 \sin\omega\tau + \omega(f_{x_\tau^\cdot}')_0 \cos\omega\tau]\, dt \tag{3.12}$$

Statement 2. Conditions (3.11) are sufficient for the asymptotic stability of the (n/m)-resonance solution of system (1.3) when $\varepsilon > 0$ is fairly small. If at least one inverse inequality holds, the solution is unstable.

The critical case of $\Delta(\tau) = 0$ is excluded by condition (3.6). When $\Delta = 0$ additional investiations are required of the conditions of existence of T-periodic solutions, which are generally associated with the expansion in fractional powers of the parameter ε. However, when the quantity $\delta(\tau) = 0$ is defined from (3.12), the stability is established taking into acocunt the higher powers of ε in the calculation of the critical characteristic exponents /1, 2/.

4. The special case ($\omega = 0$).

System (1.3) may have, when $\omega = 0$, solutions that define either oscillatory or rotational motions. In the case of rotations, the function f is, in addition, required to be 2π-periodic in x, x_τ. Such motions can occur in periodic force fields subjected to high-frequency perturbations /1, 10/.

In fact, suppose we consider an oscillatory or rotational system with one degree of freedom of the general form /8, 10/

$$q'' = Q(\Omega s, q, q', q_\sigma, q_\sigma')$$

where $q = q(s)$ is the generalized coordinate, the prime dindicates a derivative with respect to the argument s, $q_\sigma = q(s - \sigma)$, and σ is the argument lag. The following relation is assumed to be valid:

$$\Omega^{-2} Q(t, x, \Omega x^{\cdot}, x_\tau, \Omega x_\tau^{\cdot}) = \varepsilon f(t, x, x^{\cdot}, x_\tau, x_\tau^{\cdot})$$

where ε is the small parameter, $t = \Omega s$ is the new argument, $x(t) = q(s)$ is the generalized coordinate, and $\tau = \Omega\sigma$ is the lag of argument t. Then in the new variables t and x the system considered here takes the form (1.3), where $\omega = 0$.

Quasisteady oscillations. The solution is derived in the form (3.1), where $\omega = 0$, i.e. $x_0 = b = \text{const}$, and the T_ν-periodic function y is defined by (3.2). The sufficient conditions of the existence and uniqueness of the periodic solution, when $\varepsilon > 0$ is fairly small reduce to the requirement of the solvability of the equation for unknown b and of simplicity of root b^*, i.e.

$$Q(0, b) \equiv Q_0(b) = 0, \quad Q_0'(b^*) \neq 0 \tag{4.1}$$

The successive approximations are derived is the same way as in Sect.3 with $\omega = 0$. In particular

$$y_0 = \beta_0 + \alpha_0^* t + \int_0^t (t - s) f_0^* \, ds, \quad \alpha_0^* = -\frac{1}{T_\nu}\int_0^{T_\nu} (T_\nu - s) f_0^* \, ds \tag{4.2}$$

The constant β_0 is determined by the condition of periodicity of y_1 which has the form of the second equation of (3.5) as $\omega \to 0$. The proof of the method of successive approximations is similar to that in /8/. Thus the unknown T_ν-periodic motion $x(t, \varepsilon)$ is close to the stationary point $x = b^* + \varepsilon y$.

Rotational motions. The perturbed solution that corresponds to combination resonance n/m is derived in the form /8, 10/

$$x = (n/m)\nu t + b + \varepsilon y, \quad y(t + T) \equiv y(t), \quad T = mT_\nu \tag{4.3}$$

The conditions of existence and uniqueness of the T-periodic solution of (3.2) when $(\omega = 0)$ have the form (4.1). Further calculations are similar to those presented in Sect.3. The questions of proof are investigated using /1, 7–9/ as the basis. Additional investigations are required when $Q_0'(b^*) = 0$ or $Q_0 \equiv 0$; they are similar to those carried out for systems without a deviating argument /1, 8, 10/.

Investigation of stability. As noted in Sect.1, the critical case considered here belongs to a special one: one group of periodic solutions corresponds to a double zero characteristic exponent when $\varepsilon = 0$ /1/. Using reasoning similar to that applied to conventional systems, and the results in /2/, we can establish that these two exponents are of order $\sqrt{\varepsilon}$. By expanding the characteristic exponents and using the solutions of the variational equation (2.5) of the form

$$\lambda = \sqrt{\varepsilon}\lambda_1 + \varepsilon\lambda_2 + O(\varepsilon^{3/2}), \quad u = u_0 + \sqrt{\varepsilon}u_1 + \varepsilon u_2 + \varepsilon^{3/2}u_3 + O(\varepsilon^2) \tag{4.4}$$

we obtain from the T-periodicity of the functions u_0, u_1, u_2, u_3 expressions for the unknown coefficients λ_1, λ_2

$$\lambda_1^2 = T^{-1} Q_0'(b^*) \tag{4.5}$$
$$\lambda_2 = \frac{1}{2T}\int_0^T [(f_{x^{\cdot}}')_0 - (f_{x_\tau}')_0\tau + (f_{x_\tau^{\cdot}}')_0]\, dt$$

Statement 3. It follows from (4.4) and (4.5) that the periodic (quasisteady or rotational) motion of a system with a lag (1.3), when $\omega = 0$, is asymptotically stable exponentially for fairly small $\varepsilon > 0$, if $Q_0'(b^*) < 0$ (the necessary condition), $\lambda_2 < 0$, and unstable otherwise.

The equation $Q_0'(b^*) = 0$ is excluded by condition (4.1). If $\lambda_2 = 0$, additional investigation, related to the more precise determination of λ from the conditions of periodicity of the coefficients $u_4, u_5, \ldots,$ is required. This leads to an increase in the requirement for the smoothness of the function f.

5. Examples.

We shall consider specific expressions for the perturbing function f in (1.3) and investigate the conditions of existence, uniqueness and stability of periodic motion in the resonance case.

A linear system. Let

$$f = \varphi + gx + hx^{\cdot} + \chi x_\tau + \varkappa x_\tau^{\cdot} \tag{5.1}$$

where $\varphi, g, h, \chi, \varkappa$ are T_ν-periodic functions of t. The conditions of existence and uniqueness of T-periodic solutions consists of the non-degeneracy of the defining linear equation relative to the unknown parameters a, b $(\theta = \omega\tau)$ (see (3.1) and (3.4)).

$$\varphi_s = [-g_{ss} - \omega h_{sc} - \chi_{ss}\cos\theta + \chi_{sc}\sin\theta - \omega(\varkappa_{sc}\cos\theta + \varkappa_{ss}\sin\theta)]\,a + [-g_{sc} + \omega h_{ss} - \chi_{sc}\cos\theta - \chi_{ss}\sin\theta + \omega(\varkappa_{ss}\cos\theta - \varkappa_{sc}\sin\theta)]\,b$$
$$\varphi_c = [g_{cs} + \omega h_{cc} + \chi_{cs}\cos\theta - \chi_{cc}\sin\theta + \omega(\chi_{cc}\cos\theta - \varkappa_{cs}\sin\theta)]\,a + [g_{cc} - \omega h_{cs} + \chi_{cc}\cos\theta + \chi_{cs}\sin\theta - \omega(\varkappa_{cs}\cos\theta - \varkappa_{cc}\sin\theta)]\,b$$

where coefficients of type $\varphi_s, \varphi_c, g_{ss}, g_{sc}, g_{cc}$ and others are obtained by integration in the interval $t \in [0, T]$ of the functions from (5.1) $\varphi(t), g(t)$ and others, multiplied by $\sin^k \omega t \cos^l \omega t$; the powers $k, l = 0, 1, 2$ are determined by the subscripts.

If the determinant of matrix of the linear system (5.2) is $\omega\Delta(\tau) < 0$, the T-periodic solution is asymptotically stable, when $\delta(\tau) < 0$, i.e.

$$T^{-1}\delta(\tau) = \omega h_0 - \chi_0 \sin\theta + \omega\varkappa_0\cos\theta < 0 \tag{5.3}$$

where $h_0, \chi_0, \varkappa_0$ are the mean values of the functions $h, \chi, \varkappa$. In particular, if in (5.1) $g = h = \varkappa \equiv 0$, the sufficient conditions of existence, uniqueness, and asymptotic stability of the n-resonance solution have the form

$$\Delta(\tau) = \chi_{sc}^2 - \chi_{ss}\chi_{cc} < 0, \quad T^{-1}\delta(\tau) = -\chi_0\sin\theta < 0 \tag{5.4}$$

In another special case, when $g = h \equiv 0$ and $\chi, \varkappa = \text{const}$, these conditions have the form

$$\omega\Delta(\tau) = -(\chi\cos\theta + \omega\varkappa\sin\theta)^2 - (\chi\sin\theta - \omega\varkappa\cos\theta)^2 < 0$$
$$T^{-1}\delta(\tau) = -\chi\sin\theta + \omega\varkappa\cos\theta < 0 \tag{5.5}$$

Note that when $\chi^2 + \varkappa^2 > 0$ in formula (5.5), the strict inequality is satisfied by Δ for all τ. The expression for Δ, δ in (5.3)–(5.5) are represented as functions of the parameter τ to allow for a comparison with respective systems without a lagging argument.

The equation of the Duffing type. Let

$$f = f_0 \sin\nu t + dx^3 + kx_\tau \quad (f_0, \nu, d, k = \text{const}) \tag{5.6}$$

Let us investigate the conditions of existence, uniqueness, and stability of the basic resonance solution $\omega = \nu$ of system (1.3), (5.6). Equations of the type (3.4) that determine the constants a and b contain five parameters and have the form of cubic equations $(r^2 = a^2 + b^2)$

$$f_0 + {}^3/_4 dar^2 + k(a\cos\omega\tau + b\sin\omega\tau) = 0$$
$${}^3/_4 dbr^2 + k(-a\sin\omega\tau + b\cos\omega\tau) = 0 \tag{5.7}$$

When $\tau = 0$, system (5.7) is identical with the thoroughly investigated system that defines the Duffing equation /1/. By a simple transformation, system (5.7) can be reduced to a cubic equation in r^2 which contains only two dimensionless parameters and is representable in the following form, convenient for graphical investigation:

$$A = [(\gamma A + \cos\theta)^2 + \sin^2\theta]^{-1} \equiv \Phi(A, \gamma, \theta)$$
$$A = k^2 f_0^{-2} r^2 \geqslant 0, \quad \gamma = {}^3/_4 dk^{-3} f_0^2, \quad \theta = \omega\tau \tag{5.8}$$

Analysis of the set of intersection points of the ray $A \geqslant 0$ with the two-parameter sets of curves Φ shows that for $|\gamma| < \infty$ and $\theta \in [0, \pi] \pmod{\pi}$ there are one or three roots $A_1 \leqslant A_2 < A_3$, and the roots A_1 and A_2 may be the same for some values of γ, θ. For example, this is true for $\cos\theta = \pm 1$, $\gamma = \mp {}^4/_{27}$, when $A_1 = A_2 = {}^9/_4$. When $\cos\theta = 1$, $0 > \gamma > -{}^4/_{27}$ or $\cos\theta = -1$, $0 < \gamma < {}^4/_{27}$, (5.8) has three different roots: $A_1 < A_2 < A_3$. Generally the property of coincidence of the roots $A_1 = A_2$ is determined by the relation (5.8) and the condition $\Phi_A' = 1$ which lead to a quadratic equation in $z = \gamma A + \cos\theta$. The discriminant of this equation is $D = 4(1 - 4\sin^2\theta)$, hence multiple roots may occur for values of θ that satisfy the inequality $\sin^2\theta \leqslant {}^1/_4$. The respective values of γ and $A_{1,2} = A$ are

$$\gamma = \frac{1}{27}\left[\left(\cos\theta \pm \frac{1}{2}D^{1/2}\right)^2 + 9\sin^2\theta\right]\left(-2\cos\theta \pm \frac{1}{2}D^{1/2}\right)^{-1}$$
$$A = 9\left(-2\cos\theta \pm \frac{1}{2}D^{1/2}\right)^2\left[\left(\cos\theta \pm \frac{1}{2}D^{1/2}\right)^2 + 9\sin^2\theta\right]^{-1} \tag{5.9}$$

To derive the set of roots $A_j(\gamma, \theta)$ it is convenient to solve the quadratic equation (5.8) for $\gamma, |\gamma| < \infty$

$$\gamma = -w\cos\theta \pm w(w - \sin^2\theta)^{1/2}, \quad w = A^{-1} > 0 \tag{5.10}$$

It follows from (5.10) that for any θ a solution w exists, and $w \geqslant \sin^2\theta$. The boundary Γ of the set of admissable values of w for given θ is determined in the plane of the parameters (γ, w) by the equation $w = \sin^2\theta$ and (5.10). As a result we have

$$\Gamma = \mp w(1 - w)^{1/2}, \quad 0 < w \leqslant 1 \tag{5.11}$$

The set of curves $\gamma(w, \theta_i)$, $0 \leqslant \theta_i \leqslant \pi/2$ for $\theta_1 = 0$, $\theta_2 = \arccos 0.95$, $\theta_3 = \pi/6$, $\theta_4 = \arccos {}^2/_3$, $\theta_5 = \arccos 0.4$, $\theta_6 = \pi/2$ and curve $\Gamma(w)$ are shown in Fig.1 in conformity with (5.10) and (5.11). Curves for $\theta \in [\pi/2, \pi]$ are obtained by reflection from the abscissa axis which completes the construction of the set of curves required. Note that the behaviour of curves $\gamma(w, \theta)$ for large $w, w \gg 1$ (for small A, $0 < A \ll 1$), according to (5.10), is determined by the approximate formula

$\gamma \approx -w\cos\theta \pm w^{3/2}$. An important qualitative property of the set of curves $\gamma(w,\theta)$ is that at points $\gamma \in \Gamma$ their tangents are vertical, i.e. the derivatives $\partial\gamma/\partial w$ are infinite. The multiplicity of the roots A_j indicated above is related to the behaviour of the curves $\gamma(w,\theta)$ when $\sin^2\theta \leqslant {}^1/_4$. The limit multiple value of w corresponding to $\sin^2\theta = {}^1/_4$ is equal to $w = {}^1/_3$ (curve 3 in Fig.1).

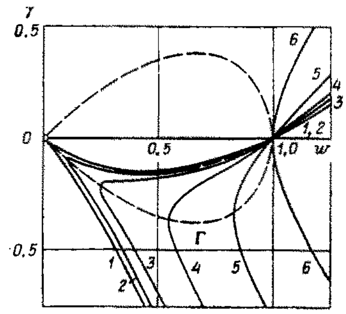

Fig.1

The functions $\gamma(w,\theta)$ provide the solution of the problem of the existence, uniqueness of the basic resonance motion of system (1.3), (5.6) and enable one to derive it, without further calculations with an error of $O(\varepsilon)$ for $t \in [0,\infty)$, if w^* is a simple root. When w^* is known, the quantity $r^2 = f_0^2(w^*k^2)^{-1}$, and the unknown parameters a and b are uniquely determined from the linear system (5.7). It will be readily seen that conditions $P = Q = 0$, $\Delta(\tau) = 0$ and the multiplicity of the roots w^*, i.e. $\Phi = A$, $\Phi_A' = 1$ $(A = w^{-1})$ are equivalent and satisfied only when $\sin^2\theta \leqslant {}^1/_4$.

The determination of stability consists of checking the conditions

$$w_j^{*2}(\gamma,\theta) + 4\gamma w_j^*(\gamma,\theta)\cos\theta + 3\gamma^2 > 0, \quad k\sin\theta > 0 \tag{5.12}$$

Unlike the motions of a classical Duffing oscillator $(\theta = 0)$ /1/, we have in system (1.3) (5.6) asymptotically stable resonance oscillations also when the second of inequalities (5.12) is satisfied. It should be noted that for the roots $w_j^*(\gamma,\theta)$ of both branches of curves $\gamma(w,\theta)$ (5.10) the verification that the first of conditions of (5.12) is satisfied reduces to checking the following inequality:

$$3w_j^* \mp 2(w_j^* - \sin^2\theta)^{1/2}\cos\theta - 2\sin^2\theta > 0$$

This and the set of curves represented in Fig.1 shows that motions which correspond to roots A_1 and A_3 for $\gamma < 0$ or A for $\gamma > 0$ are asymptotically stable, when θ is small if $k > 0$. The stability of fundamental resonance oscillations when $\theta \in [\pi/2, \pi]$ is analysed similarly; the case when $\theta \in [\pi, 2\pi]$ is the same as that considered here.

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REFERENCES

1. MALKIN I.G., Some Problems of the Theory of Non-linear oscillations. Moscow, Gostekhizdat, 1958.
2. SHIMANOV S.N., On the stability of quasiharmonic systems with a lag. PMM, Vol.25, No.6, 1961.
3. EL'SGOL'TS L.E. and NORKIN S.B., Introduction to the Theory of Differential Equations with Deflecting Argument. Moscow, Nauka, 1971.
4. KRASOVSKII N.N., On the optimal control of linear systems with a time lag. Sib. Matem. Zh., Vol.4, No.2, 1963.
5. RUBANIK V.P., Oscillations of Quasilinear Systems with lag. Moscow, Nauka, 1969.
6. KRASOVSKII N.N., The periodic solutions of differential equations with a time lag. Dokl. AN SSSR, Vol.114, No.2, 1957.
7. SHIMANOV S.N., On the theory of oscillations of quasilinear systems with lag. PMM Vol.25, No.5, 1959.
8. AKULENKO L.D., Steady periodic and rotary motions of perturbed essentially non-linear systems with one degree of freedom, close to conservative, in the case of an arbitrary constant deviation of the arguments. PMM Vol.32, No.1, 1968.
9. KOLLATZ L., Functional analysis and Computational Mathematics, Moscow, Mir, 1969.
10. AKULENKO L.D., On certain rotational-oscillatory systems subjected to high-frequency perturbations. Zh. Vychisl. Matem. i Matem. Fiz., Vol.8, No.5, 1968.

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